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Book Review: *The Self-Avoiding Walk*

The Self-Avoiding Walk, N. Madras and G. Slade, Birkhäuser, Boston, 1993, xiv + 425 pp. (hard cover 1993, soft cover 1996).

From the preface: "A self-avoiding walk (SAW) is a path on a lattice that does not visit the same site more than once. In spite of this simple definition, many of the most basic questions about this model are difficult to resolve in a mathematically rigorous fashion."

This monograph is an up to date—and essentially complete—account of the mathematically rigorous results that were known by 1993 for SAWs on lattices, typically $\mathbb{Z}^d$ ($d \geqslant 1$). In addition, it describes a variety of Monte Carlo algorithms that have been used to simulate SAWs, and lists results that have been obtained via exact enumeration. It also includes a brief description of heuristic methods coming from physics and chemistry, in particular, scaling theory. Not included are renormalization group methods and conformal invariance in two dimensions. However, this would have made the monograph too long.

The monograph is written for probabilists, mathematical physicists and combinatorialists. A prerequisite for reading is a knowledge of elementary probability and analysis, and some background in statistical physics. The style of the monograph is such that it systematically builds up the mathematical theory of SAW while letting itself be driven by ideas coming from statistical physics, in particular, the theory of critical phenomena. Much care is taken to constantly provide the reader with this context and motivation, which makes for very pleasant reading. The text is well organized, there are historical notes at the end of every chapter, and there is an exhaustive list of references. It is nice to see the panorama of SAW develop as one goes along, and a good deal of nice applied probability is encountered along the way.

SAW is closely linked with other models from statistical physics, like the Ising and Potts model, percolation, lattice trees and lattice animals. In all these models scaling theory and critical exponents are among the key

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properties to be investigated. SAW is the simplest example in this class of models and therefore serves as a kind of paradigm.

Chapter 1 gives an outline of the basic questions in the area, which concern the following quantities:

(1) $c_n(0, x)$, the number of n-step SAWs beginning at 0 and ending at x;

(2) $c_n = \sum_x c_n(0, x)$, the number of n-step SAWs beginning at 0;

(3) $\mu = \lim_{n \to \infty} c_n^{1/n}$, the connective constant;

(4) $G_z(0, x) = \sum_{n \geqslant 0} z^n c_n(0, x)$ $(|z| \leqslant z_c = 1/\mu)$, the two-point function;

(5) $\chi(z) = \sum_{n \geqslant 0} z^n c_n$ $(|z| \leqslant z_c)$, the susceptibility;

(6) $\xi(z) = [-\lim_{m \to \infty} (1/m) \log G_z(0, me)]^{-1}$ $(|e| = 1)$, the correlation length;

(7) $\langle |\omega(n)|^2 \rangle = (1/c_n) \sum_x |x|^2 c_n(0, x)$, the mean-square displacement of a typical n-step SAW.

A list is made of the conjectures for the various critical exponents that are believed to characterize the asymptotic behavior of these quantities for $n \to \infty$, $|x| \to \infty$ or $z \uparrow z_c$. These conjectures are part of the folklore of the area, and some of them are proved later on. There is also a discussion of the role of the "bubble diagram" $B(z) = \sum_x [G_z(0, x)]^2$, showing that $B(z_c) < \infty$ signals mean-field behavior. In this chapter the stage is set for the rest of the monograph.

Chapter 2 describes non-rigorous ideas from scaling theory, as it applies to a variety of models of which SAW is an example. There is a description of Flory's heuristic argument identifying the critical exponent in the conjecture $\langle |\omega(n)|^2 \rangle \sim Dn^{2\nu}$ $(n \to \infty)$ as $\nu = \max\{\frac{1}{2}, 3/(d+2)\}$, which is known to be correct for $d = 1$ and $d > 4$, believed to be correct for $d = 2$ and $d = 4$ (modulo a logarithmic correction), and known to be wrong for $d = 3$. Furthermore, there is a discussion of the connection, first discovered by de Gennes, that SAW arises as the $N \to 0$ limit of an N-vector model, where spins take values in the N-dimensional sphere of radius $\sqrt{N}$ and interact via a Hamiltonian summing the inner products of nearest-neighbor spins. This connection is pivotal for much of the scaling theory.

Chapter 3 derives upper and lower bounds on c_n for $\mathbb{Z}^d$ $(d \geqslant 2)$ due to Hammersley–Welsh and Kesten. For $d = 2$, 3 and 4 these bounds are still the best rigorous results known to date. The proofs all make use of the notion of subadditivity, in combination with various geometrical operations, such as concatenating, cutting or reflecting SAWs. Related results are described for self-avoiding bridges (SABs) and self-avoiding polygons (SAPs).

Chapter 4 contains a detailed study of $G_z(0, x)$. In particular, for $|z| < z_c$ a proof is given of Ornstein–Zernike decay as x moves out along a lattice axis. The proof proceeds by first deriving the result for SABs and then extending to SAWs. A key tool is a renewal type argument for SABs.

Chapter 5 develops the "lace expansion" for SAW, originally introduced by Brydges and Spencer. This is a diagrammatic technique in which the two-point function is expanded in terms of certain irreducible loop diagrams, in the spirit of cluster expansions in statistical physics. Two ways to obtain the lace expansion are described: (i) via the inclusion-exclusion principle; (ii) via an algebraic calculation involving weight factors. The goal in this chapter is to set up the machinery that is used in Chapter 6 to prove mean-field behavior of SAW for $d > d_c$, with $d_c = 4$ the upper critical dimension. The issue of convergence of the expansion is deferred to Chapter 6. Also derived are bounds on the lace expansion in terms of the two-point function, which play a key role in Chapter 6 to establish convergence of the lace expansion for $d > d_c$. These bounds are in fact valid more generally for models with a "repulsive" interaction. To illustrate the general context, the lace expansion is also set up for lattice trees, lattice animals and percolation, although these are not used later in the monograph. Over the past decade or so, the lace expansion method has emerged as the main tool for establishing mean-field behavior in a variety of models above their upper critical dimension. Most of the material discussed in Chapters 5 and 6 is based on work of Hara and Slade.

Chapter 6 uses the lace expansion for SAWs that was set up in Chapter 5 to derive mean-field behavior for: (I) nearest-neighbor SAW in sufficiently high dimensions; (II) "sufficiently spread out" SAW in any dimension $d > 4$ (where spread out means that the steps of the SAW are not restricted to nearest-neighbors but to some finite box). In particular, the convergence issue is settled here. A whole series of results is discussed, showing that for both (I) and (II) the critical exponents exist and assume their mean-field value as predicted by the heuristic scaling theory. In a way, this chapter is the apotheosis of the monograph, showing that all the basic results that are believed to be true in arbitrary dimension are actually rigorously true above the upper critical dimension $d_c = 4$. Hara and Slade have given a computer-assisted proof of the convergence of the lace expansion for nearest-neighbor SAW in dimension $d \geqslant 5$, but this is technically complex and therefore is not included.

Chapter 7 discusses Kesten's pattern theorem. A pattern is any finite SAW that occurs as part of a longer SAW. The theorem says that if a pattern can occur at least three times on a SAW, then it occurs at least an times on an n-step SAWs for all n sufficiently large and some $a > 0$ (depending on the pattern), with the exception of a subclass of the SAWs that is exponentially small in n. The proof is geometrical and uses patterns enclosed in cubes. The pattern theorem has a number of interesting consequences. For instance, it implies ratio limit theorems for c_n and its analogue for SABs and SAPs. There are strong results for the fraction of SAWs in which a certain pattern occurs either at the very beginning or at the very end. These are used in analyzing the behavior of certain Monte Carlo algorithms.

Chapter 8 describes a potpourri of results: (a) upper bounds on $c_n(0, x)$ for SAWs and SABs with the help of the renewal type argument in Chapter 4; (b) analysis and comparison of the connective constants for SAWs, SABs and SAPs in confined geometries (i.e., subsets of $\mathbb{Z}^d$ like tubes, slabs or wedges); (c) construction of the "infinite" SAB; (d) a proof that "unknotted" SAPs form an exponentially small subclass (i.e., long SAPs are typically intertwined a lot). Most of these results make use of Kesten's pattern theorem. The infinite SAB is constructed as follows. Given a finite pattern ω, let $p_n(\omega)$ denote the fraction of SABs of length n that begin with ω. It is shown that $\lim_{n \to \infty} p_n(\omega) = p(\omega)$ exists for all ω. This defines a probability measure $p(\cdot)$ on finite SABs, which is consistent and therefore uniquely defines a probability measure on infinite SABs. For SAWs the convergence is known only for cases (I) and (II) in Chapter 6.

Chapter 9 is a self-contained mini-review of Monte Carlo algorithms that have been used to simulate SAWs. It opens with some simple examples that serve to illustrate the main questions and problems one has to face, and describes some statistical theory one needs in order to handle the data properly. The rest of the chapter describes a variety of algorithms, with special emphasis on rigorous analysis of ergodicity and performance quality: (i) static methods, where the algorithm generates a single SAW with prescribed properties; (ii) dynamic methods, where the algorithm generates a Markov chain whose equilibrium is a prescribed probability measure on a class of SAWs. A comparison is made between algorithms that make local vs. global moves, generate SAWs of fixed vs. variable length, and fixed vs. variable endpoint. Typically it is hard to obtain rigorous results about the performance quality, but some are proved here.

Chapter 10 discusses a miscellany of related topics: (a) weakly self-avoiding walk (where intersections are not forbidden but are discouraged) and its Brownian analogue; (b) loop-erased random walk (obtained from an infinite random walk path by successively erasing loops); (c) intersection properties of two or more simple random walks; (d) the "true" SAW (a random process with transition rates that discourage self-intersections). Models (b) and (d) have a behavior that is different from SAW, though very reminiscent. Model (a), on the other hand, falls in the same universality class.

The monograph closes with three appendices on simple random walk, renewal theory and exact enumeration, respectively.

Since 1993, a number of interesting developments have taken place. The most important of these are:

1. Implementation of the lace expansion for lattice trees, directed percolation, and percolation, leading to a proof of mean-field behavior above the upper critical dimensions for these models.

2. Complete analysis of weakly SAW and "true" SAW in dimension $d=1$ with the help of large deviation methods.

3. More refined analysis of intersection exponents for two or more simple random walks.

4. Improved results from Monte Carlo algorithms and exact enumeration. Reference: S. G. Whittington (editor), *Numerical Methods for Polymeric Systems*, The IMA Volumes in Mathematics and its Applications 102, Springer, New York, 1998.

5. Further refined predictions for two-dimensional SAWs via conformal invariance. Results for SAWs with self-attraction and for SAWs interacting with a surface. Reference: C. Vanderzande, *Lattice Models of Polymers*, Cambridge Lecture Notes in Physics 11, Cambridge University Press, Cambridge, 1998.

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